Fall 2024

Question 3 : let $\underline{b} = \{b_1, b_2, b_3\}$ bases of $\mathbb{R}^3$

$\underline{c} = \{c_1, c_2, c_3\}$

related by

$$c_1 = b_1 - b_2$$
$$c_2 = b_1 - b_3$$
$$c_3 = b_1 + b_2 + b_3$$

Let P be the c.o.b. matrix $P[v]_{\underline{b}} = [v]_{\underline{c}}, \forall v \in \mathbb{R}^3$.

$P_{23} = 1$ $\qquad$ $P_{23} = 2$ $\qquad$ $P_{23} = -\frac{2}{3}$ $\qquad$ $P_{23} = 0$

$$P = \begin{pmatrix} * & * & * \\ * & * & P_{23} \\ * & * & * \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} * \\ P_{23} \\ * \end{pmatrix}$$

$$Pe_3 = \ldots\ldots + P_{23} e_2 + \ldots\ldots$$

Choose $v = b_3$ in the formula $P[v]_{\underline{b}} = [v]_{\underline{c}}$, because

$[b_3]_{\underline{b}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = e_3$; with this choice, $[b_3]_{\underline{c}} = ?$

eliminate b_1, b_2 from RHS $\Leftarrow$

$$c_1 = b_1 - b_2$$
$$c_2 = b_1 - b_3$$
$$c_3 = b_1 + b_2 + b_3$$

$b_1 = c_1 + b_2$

$$\begin{cases} c_2 = c_1 + b_2 - b_3 \\ c_3 = c_1 + 2b_2 + b_3 \end{cases} \Rightarrow \begin{cases} b_2 - b_3 = c_2 - c_1 \\ 2b_2 + b_3 = c_3 - c_1 \end{cases} \Rightarrow \begin{cases} 2b_2 - 2b_3 = 2c_2 - 2c_1 \\ 2b_2 + b_3 = c_3 - c_1 \end{cases} \; (-)$$

$$-3b_3 = 2c_2 - c_1 - c_3$$

$$\Downarrow$$

$$b_3 = \frac{c_1}{3} - \frac{2}{3}c_2 + \frac{c_3}{3}$$

$$Pe_3 = [b_3]_{\underline{c}} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$Pe_3 = \ldots\ldots + p_{23}e_2 + \ldots\ldots$$

$$\Downarrow$$

$$p_{23} = -\frac{2}{3}.$$

Question 6: $A \in \mathbb{R}^{2\times 2}$;

eigenvalues $\rightsquigarrow$ eigenvectors

$\lambda_1 = 1$ $\qquad$ $v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\lambda_2 = 2$ $\qquad$ $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$A^{-1} = ?$$

Method 1 (brute force): calculate A and invert

$$A = PDP^{-1} \quad , \quad P = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \rightsquigarrow P^{-1} = \frac{1}{2\cdot 2 - 1\cdot 3}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 2 \\ -6 & 5 \end{pmatrix} \rightsquigarrow A^{-1} = \frac{1}{(-2)5 - 2(-6)}\begin{pmatrix} 5 & -2 \\ 6 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{2} & -1 \\ 3 & -1 \end{pmatrix}$$

Method 2: $A = PDP^{-1} \rightsquigarrow A^{-1} = (P^{-1})^{-1} D^{-1} P^{-1} = P D^{-1} P^{-1}$

$$A^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{1} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & \frac{1}{2} \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{2} & -1 \\ 3 & -1 \end{pmatrix}.$$

Question 20 (T/F).

if $A \in \mathbb{R}^{m \times n}$ s.t. $Col(A) = \mathbb{R}^m$, then for every $b \in \mathbb{R}^m$ the linear system $Ax = b$ has a unique solution

$Col(A) = \{ b \in \mathbb{R}^m \mid Ax = b$ has a solution$\}$

$= \mathbb{R}^m$ if and only if $Ax = b$ has at least one solution for all $b \in \mathbb{R}^m$

$Ker(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$

$= \{0\}$ if and only if $Ax = b$ has at most one solution for all $b \in \mathbb{R}^m$

F; because uniqueness means both $Col(A)=\mathbb{R}^m$ and $Ker(A)=\{0\}$

and $\exists$ matrices with $Col(A)=\mathbb{R}^m$ but $Ker(A)\neq\{0\}$

$(2\ \ 3) \in \mathbb{R}^{1\times 2}$; $m=1$; $Col(A)=\mathbb{R}$

$Ker(A)=\left\{\begin{pmatrix}3\\-2\end{pmatrix}t \;\middle|\; t\in\mathbb{R}\right\}$

Rank-nullity : $\dim Col(A) + \dim Ker(A) = n$

if $n>m$ and $Col(A)=\mathbb{R}^m$, then $Ker(A)\neq\{0\}$

Question 26 (T/F): $A,B\in\mathbb{R}^{n\times n}$

if A,B are symmetric, then $A+B$ is diagonalizable

$\Downarrow$

$A+B$ is symmetric $\Rightarrow$ sym means orthogonally diagonalizable

T

$$\begin{pmatrix} a_{11} & x \\ x & a_{22}\end{pmatrix} + \begin{pmatrix} b_{11} & y \\ y & b_{22}\end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & x+y \\ x+y & a_{22}+b_{22}\end{pmatrix}$$

$(A+B)^T = A^T + B^T = A + B \Rightarrow A+B$ symmetric

Question 30: $A \in \mathbb{R}$, $A^3 = I_n$

(a) find $\text{Ker}(A)$

(b) let $\lambda \in \mathbb{R}$ be an eigenvalue of A. Compute λ.

(a): pick an arbitrary $v \in \text{Ker}(A) \iff Av = 0$

$$Av = 0 \implies AAAv = 0 \implies I_n v = 0 \implies v = 0$$

Upshot: $\text{Ker}(A) \subseteq \{0\}$

$\text{Ker}(A) \supseteq \{0\}$ always true b/c $\text{Ker}(A)$ is a subspace and subspaces contain 0 vector

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$$\text{Ker}(A) = \{0\}$$

(b): λ is eigenvalue of A means $Av = \lambda v$ for some $v \neq 0$

but $A^3 = I_n$

$$v = I_n v = A^3 v = AAAv = AA\lambda v$$
$$= \lambda AAv = \lambda A \lambda v$$
$$= \lambda^2 A v = \lambda^3 v$$

$\lambda = 1$ b/c λ is real $\Longleftarrow$ $\lambda^3 = 1$ because $v \neq 0$

(if eigenvalues of A are $\lambda_1, \dots, \lambda_n$, then eigenvalues of A^k are $\lambda_1^k, \dots, \lambda_n^k$)
for all $k \in \mathbb{Z}$

Question 31:
$$A = \begin{pmatrix} 2025 & 1 & 1 & 1 \\ 1 & 2025 & 1 & 1 \\ 1 & 1 & 2025 & 1 \\ 1 & 1 & 1 & 2025 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(a) Show that v is an eigenvector for A and find eigenvalue

(b) Find a basis for the eigenspace of $\lambda = 2024$

(c) Find a diagonal matrix D similar to A

(a) $Av = \mu v$ for some μ to be calculated

$$\begin{pmatrix} 2025 & 1 & 1 & 1 \\ 1 & 2025 & 1 & 1 \\ 1 & 1 & 2025 & 1 \\ 1 & 1 & 1 & 2025 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \mu \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2028 \\ 2028 \\ 2028 \\ 2028 \end{pmatrix} = \mu \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$; choose $\mu = 2028$ and this computation shows that $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \neq 0$ is a eigenvector for $\mu = 2028$

$$V_{2028} = \{ w \text{ s.t. } Aw = 2028\, w \} \underline{\underline{\supseteq}} \operatorname{span} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$ 1-dim

$\overset{\shortparallel}{v_4}$

(b) $V_{2024} = \{ w \text{ s.t. } Aw = 2024w \}$

$= \operatorname{Ker}(A - 2024 \cdot I_4)$

$= \operatorname{Ker}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

$= \operatorname{Ker}\begin{pmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$= \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \text{ s.t. } \underset{\substack{\Updownarrow \\ x = -y-z-t}}{x+y+z+t = 0} \right\}$

$= \left\{ \begin{pmatrix} -a-b-c \\ a \\ b \\ c \end{pmatrix} \text{ s.t. } a, b, c \in \mathbb{R} \right\}$

$= \operatorname{Span}\left\{ \underset{\substack{\shortparallel \\ v_1}}{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{\substack{\shortparallel \\ v_2}}{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{\substack{\shortparallel \\ v_3}}{\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$ 3-dim

2024 and 2028 are only eigenvalues

(c) part (a) tells us that geom mult$_{2028} \geq 1$

part (b) tells us that geom mult$_{2024} = 3$

but alg mult$_\lambda \geq$ geom mult$_\lambda$ for all λ

and $\sum$ alg mult$_\lambda$ = 4 $\geq \sum$ geom mult$_\lambda \geq 1+3+\ldots$

inequalities force

- geom $\text{mult}_{2028} = 1$
- alg mult_{2024} = geom mult_{2024}

  alg mult_{2028} = geom mult_{2028}
- no other eigenvalues

$\Downarrow$

$\text{Spec}(A) = \{2024, 2024, 2024, 2028\}$ and geom mult = alg mult for every eigenvalue

$\Downarrow$

A is diagonalizable, i.e. $A = PDP^{-1}$ with $D = \begin{pmatrix} 2024 & & & 0 \\ & 2024 & & \\ & & 2024 & \\ 0 & & & 2028 \end{pmatrix}$

$\Downarrow$

$$A \sim D = \begin{pmatrix} 2024 & & & 0 \\ & 2024 & & \\ & & 2024 & \\ 0 & & & 2028 \end{pmatrix}$$

$$\left(P = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix} \right)$$